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DECAY OF SOLUTIONS OF THE WAVE EQUATION WITH A LOCAL DEGENERATE DISSIPATION

BY

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ABSTRACT

We derive a precise decay estimate of the solutions to the initial-boundary value problem for the wave equation with a dissipation:

$$u_{tt} - \Delta u + a(x)u_t = 0 \quad \text{in } \Omega \times [0, \infty)$$

with the boundary condition $u|_{\partial\Omega} = 0$, where $a(x)$ is a nonnegative function on $\overline{\Omega}$ satisfying

$$a(x) > 0 \text{ a.e. } x \in \omega \text{ and } \int_\omega \frac{1}{a(x)^p} dx < \infty \text{ for some } 0 < p < 1$$

for an open set $\omega \subset \overline{\Omega}$ including a part of $\partial\Omega$ with a specific property. The result is applied to prove a global existence and decay of smooth solutions for a semilinear wave equation with such a weak dissipation.

1. Introduction

In this paper we are concerned with the decay property of the solutions to the initial-boundary value problem for the wave equation with a dissipation:

$$u_{tt} - \Delta u + a(x)u_t = 0 \quad \text{in } \Omega \times [0, \infty) \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0), = u_1(x) \quad \text{and} \quad u|_{\partial\Omega} = 0, \tag{1.2}$$

where Ω is a bounded domain in R^N with a smooth boundary $\partial\Omega$ and $a(x)$ is a smooth nonnegative function on $\overline{\Omega}$ which may vanish somewhere in $\overline{\Omega}$.

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When $a(x) \geq \epsilon_0 > 0$ on $\overline{\Omega}$ it is easy to show that the solutions of (1.1)–(1.2) decay exponentially to 0 as $t \to \infty$, that is, we can prove for an energy finite solution $u(t)$

$$E(t) \equiv \frac{1}{2}\{\| u_t(t) \|^2 + \| \nabla u(t) \|^2\} \leq CE(0)e^{-\lambda t} \tag{1.3}$$

with some $\lambda > 0$ (for a preciser or more general result see Rauch and Taylor [15], Nakao [9] etc.).

On the other hand, if $a(x_0) > 0$ for some $x_0 \in \Omega$ it is known that

$$\lim_{t\to\infty} E(t) = 0 \tag{1.4}$$

(cf. N. Iwasaki [5], C. Dafermos [2] and A. Haraux [3]).

Recently in [11] we gave an intermediate result of (1.3) and (1.4) as follows: Assume that

$$a(x) > 0 \quad \text{a.e.} \quad x \in \overline{\Omega} \quad \text{and} \quad \int_\Omega \frac{1}{a(x)^p} dx < \infty \tag{1.5}$$

for some $0 < p < 1$. Then, it holds that

$$E(t) \leq C(\| u_0 \|_{H_{m+1}} + \| u_1 \|_{H_m})(1+t)^{-2pm/N} \tag{1.6}$$

where $m > N/2$ and $(u_0, u_1) \in H_{m+1} \times H_m$ should satisfy the compatibility condition of the mth order.

Roughly speaking, the condition on $a(x)$ in the above admits that $a(x)$ vanishes on any $N-1$ dimensional submanifolds in $\overline{\Omega}$. The estimate (1.6) means that the decay rate depends on the degeneracy of $a(x)$ as well as the reguráity of the solution itself. The method obtaining (1.6) is used in [12, 13] to prove the global existence and decay of the smooth solutions for some semilinear and quasilinear equations with such a degenerate dissipation. For a related result to (1.6) see also D. Russell [16].

Another approach was employed by Bardos, Lebeau and Rauch in [1]. There, it is proved by a micro-local analysis technique that when $\partial\Omega$ and $a(x)$ are of class C^∞, (1.3) holds if and only if the following condition is satisfied: There exists $T > 0$ such that every ray of geometric optics intersects the set $\{x \in \overline{\Omega} | a(x) > 0\} \times (0, T)$. A typical case which assures this condition is that $a(x) \geq \epsilon_0 > 0$ in a neighbourhood of $\partial\Omega$. Note that the above condition in [1] and (1.5) are independent of each other.

Quite recently, E. Zuazua [17] gave a simple sufficient condition on $a(x)$ for (1.3) to hold. That is, we set for $x_0 \in R^N$,

$$\Gamma(x_0) = \{x \in \partial\Omega | (x - x_0) \cdot \nu(x) \geq 0\} \tag{1.7}$$

where $\nu(x)$ denotes the outward unit normal of the boundary $\partial\Omega$ at $x \in \partial\Omega$ (cf. Lions [7]), and we assume that there exists $x_0 \in R^N$ and a neighbourhood ω of $\Gamma(x_0)$ in $\overline{\Omega}$ such that

$$a(x) \geq \epsilon_0 > 0 \quad \text{on } \omega. \tag{1.8}$$

Then, the estimate (1.3) holds for every energy finite solution $u(t)$.

This result due to Zuazua can be applicable to some type of semilinear equations without smallness condition on the initial data.

The object of this paper is to combine the methods in [11] and [17] to prove the following more general result: If $a(x) > 0$ a.e. $x \in \omega$, which is the same as above, and

$$\int_\omega \frac{1}{a(x)^p} dx < \infty \tag{1.9}$$

for some $0 < p < 1$, then the estimate (1.6) holds.

We further show that this result is useful to prove the global existence and decay of smooth solutions for some semilinear equations. Some semilinear and also quasilinear equations with a degenerate dissipation are treated in [12] and [13] by an energy method, but our situation is a little more delicate and we must employ another method for the global existence.

2. Preliminaries and results

We use only standard function spaces and omit their definition. But, we note that $\|\cdot\|$ denotes the L^2 norm on Ω.

If $a(x)$ is smooth and $u(t)$ is a smooth solution of the Problem (1.1)–(1.2), $\frac{\partial^k}{\partial t^k}u(t)$, $k = 0, 1, \ldots, m$, must be 0 on the boundary $\partial\Omega$. From the equation (1.1) we have, for $k \geq 2$,

$$\frac{\partial^k}{\partial t^k}u(t) = \Delta\frac{\partial^{k-2}}{\partial t^{k-2}}u(t) - a(x)\frac{\partial^{k-1}}{\partial t^{k-1}}u(t) \tag{2.1}$$

and then we define $u_k \in H_{m+1-k}$, by induction, as follows:

$$u_k = \Delta u_{k-2} - au_{k-1}, \quad k = 2, 3, \ldots, \tag{2.2}$$

where u_0, u_1 are those given as initial data.

Definition 1: We say that $(u_0, u_1) \in H_{m+1} \times H_m$ satisfies the compatibility condition of the mth order associated with (1.1)–(1.2) if

$$u_k \in H_{m+1-k} \bigcap H_1^0 \quad \text{for } k = 0, 1, \ldots, m, \text{ and } u_{m+1} \in L^2.$$

The following existence theorem is standard (cf. M. Ikawa [4], T. Kato [6], Pazy [14] etc.).

PROPOSITION 1: *Let $m \geq 0$ be an integer. Suppose that $a(\cdot)$ belongs to $C^{m-1}(\overline{\Omega})$ ($a \in L^\infty$ if $m = 0$) and $(u_0, u_1) \in H_{m+1} \times H_m$ $(H_0 = L^2)$ satisfies the compatibility condition of mth order associated with (1.1)–(1.2). Then, there exists a unique solution $u(t)$ of the problem (1.1)–(1.2) such that*

$$u \in X_m \equiv \bigcap_{k=0}^{m} C^k([0, \infty); H_{m+1-k} \cap H_1^0) \cap C^{m+1}([0, \infty); L^2(\Omega)). \tag{2.3}$$

Moreover, we have

$$\sum_{k=0}^{m+1} \| D^k u(t) \|^2 \leq C(\| u_0 \|_{H_{m+1}}^2 + \| u_1 \|_{Hm}^2), \tag{2.4}$$

where D^k denotes any partial differentiations with respect to t and x of order k and C denotes a general positive constant.

Our result on the linear equation reads as follows.

THEOREM 1: *Suppose that $a(x) \geq 0$ on Ω and there exists $x_0 \in R^N$ and a neighbourhood ω of $\Gamma(x_0)$ (see (1.7)) such that*

$$a(x) > 0 \quad \text{a.e. } x \in \omega \quad \text{and} \quad \int_\omega a(x)^{-p} dx < \infty \tag{2.5}$$

for some $0 < p < 1$. Further, suppose that $a(\cdot)$ belongs to $C^{m-1}(\overline{\Omega})$ and (u_0, u_1) satisfies the compatibility condition of mth order with m such that

$$m > N/2. \tag{2.6}$$

Then, the solution $u(t)$ of (1)–(2) in Proposition 1 meets the decay property

$$E(t) \leq \{E(0)^{-N/2mp} + C(\|u_0\|_{H_{m+1}} + \|u_1\|_{H_m})^{-N/mp}(t - T)^+\}^{-2mp/N} \tag{2.7}$$

for $0 \le t < \infty$ *with some* $T > 0$ *independent of* (u_0, u_1), *where we use the notation* $\alpha^+ = \max\{\alpha, 0\}$.

Next, we consider the semilinear equation of the form

$$u_{tt} - \Delta u + a(x)u_t + f(u) = 0, \tag{2.8}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{and} \quad u|_{\partial\Omega} = 0. \tag{2.9}$$

The compatibility condition on (u_0, u_1) associated with (2.8)–(2.9) is defined similarly as in Definition 1 through the equations for an assumed smooth solution $u(t)$:

$$\frac{\partial^k}{\partial t^k}u(t) = \Delta \frac{\partial^{k-2}}{\partial t^{k-2}}u(t) - a(\cdot)\frac{\partial^{k-1}}{\partial t^{k-1}}u(t) - \frac{\partial^{k-2}}{\partial t^{k-2}}f(u(t)) \tag{2.10}$$

$(k = 2, 3, \ldots)$.

To make the essential feature clear we restrict ourselves to the important cases $N = 2, 3$, and assume that $f \in C^4(R)$ and

$$|f(u)| \le k_0|u|^3, \quad |f'(u)| \le k_1|u|^2 \quad \text{and} \quad |f''(u)| \le k_2|u| \tag{2.11}$$

for $|u| \le L$, where $k_i, i = 1, 2, 3$, are positive constants depending on L, L being arbitrarily fixed hereafter. We note that no growth condition on $f(u)$ as $|u| \to \infty$ is made in our argument.

We prove the following result.

THEOREM 2: *Let* $N = 2, 3$ *and* $a(\cdot), p$ *satisfy the same hypotheses as in Theorem 1 with* $m = 4$. *We assume that* $(u_0, u_1) \in H_5 \times H_4$ *satisfies the compatibility condition of the* 4*th order associated with* (2.8)–(2.9) *and*

$$2mp \equiv 8p > N. \tag{2.12}$$

Then, for any $K > 0$ *there exists* $\epsilon(K) > 0$ *such that if*

$$\|u_0\|_{H_5} + \|u_1\|_{H_4} < K \quad \textit{and} \quad I_0 \equiv \|\nabla u_0\| + \|u_1\| < \epsilon(K), \tag{2.13}$$

the problem (2.8)–(2.9) *admits a unique solution* $u(t) \in X_4$ *and it holds that*

$$\sum_{k=0}^{5} \|D_t^k u(t)\|_{H_{5-K}} \le K < \infty \tag{2.14}$$

and

$$E(t) \le C(K)(1+t)^{-8p/N}, \tag{2.15}$$

where D_t^k denotes the kth order partial differentiation with respect to t.

To prove the theorems we need the following lemmas.

LEMMA 1 (Gagliardo–Nirenberg): *Let $1 \le r < p \le \infty, 1 \le q \le p$ and $m \ge 0$. Then, we have the inequality*

$$\|v\|_{W^{k,p}} \le C\|v\|_{W^{m,q}}^{\theta}\|v\|_r^{1-\theta} \quad \text{for } v \in W^{m,q} \cap L^r \tag{2.16}$$

with some $C > 0$ and

$$\theta = \left(\frac{k}{N} + \frac{1}{r} - \frac{1}{p}\right)\left(\frac{m}{N} + \frac{1}{r} - \frac{1}{q}\right)^{-1} \tag{2.17}$$

provided that $0 < \theta \le 1$ $\quad(0 < \theta < 1$ if $p = \infty$ and $mq = integer)$.

LEMMA 2: *Let $\phi(t)$ be a nonnegative function on $R^+ = [0, \infty)$ satisfying*

$$\sup_{t \le s \le t+T} \phi(s)^{1+\gamma} \le g(t)\{\phi(t) - \phi(t+T)\} \tag{2.18}$$

with $T > 0, \gamma > 0$ and $g(t)$ is a nondecreasing function. Then, $\phi(t)$ has the decay property

$$\phi(t) \le \left\{\phi(0)^{-\gamma} + \gamma \int_T^t g(s)^{-1} ds\right\}^{-1/\gamma} \quad \text{for } t \ge T. \tag{2.19}$$

For a proof of Lemma 2 see Nakao [8, 10]. If $\gamma = 0$ and $g(t)$ =const in the above we have

$$\phi(t) \le C\phi(0)e^{-\lambda t}$$

for some $\lambda > 0$.

3. Proof of Theorem 1

We give a proof of Theorem 1 by combining the techniques in [11] and [17].

Multiplying the equation by u_t and integrating over $[t, t+T] \times \Omega, t > 0, T > 0$, we have

$$\int_t^{t+T} \int_\Omega a(x)|u_t|^2 dxds = E(t) - E(t+T) \equiv D(t)^2. \tag{3.1}$$

Next, multiplying the equation by u and integrating we have

$$\begin{aligned} &\int_t^{t+T} \int_\Omega (|\nabla u|^2 - |u_t|^2) dxds \\ &= -\int_t^{t+T} \int_\Omega a u_t u dxds - (u_t(t+T), u(t+T)) + (u_t(t), u(t)). \end{aligned} \tag{3.2}$$

We shall derive the inequality

$$\int_t^{t+T} E(s)ds \le C \left\{ E(t+T) + D(t)^2 + \int_t^{t+T} \int_\omega (|u_t|^2 + |u|^2) dxds \right\}. \tag{3.3}$$

The derivation of (3.3) is essentially due to Zuazua [17], and we sketch it briefly.

Multiplying the equation by $(x - x^0) \cdot \nabla u$ and integrating we have

$$\begin{aligned} &\frac{N}{2} \int_t^{t+T} \int_\Omega (|u_t|^2 - |\nabla u|^2) dxds + \int_t^{t+T} \int_\Omega |\nabla u|^2 dxds \\ &+ \int_t^{t+T} \int_\Omega a u_t (x - x^0) \cdot \nabla u dxds = -(u_t(t+T), (x - x^0) \cdot \nabla u(t+T)) \\ &+ (u_t(t), (x - x^0) \cdot \nabla u(t)) + \frac{1}{2} \int_t^{t+T} \int_{\partial\Omega} (x - x^0) \cdot \nu \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma ds. \end{aligned} \tag{3.4}$$

It follows from (3.2) and (3.4) that

$$\begin{aligned} &\left(\frac{N}{2} - \alpha\right) \int_t^{t+T} \|u_t(s)\|^2 ds + \left(1 + \alpha - \frac{N}{2}\right) \int_t^{t+T} \|\nabla u(s)\|^2 ds \\ &\le C \left(\int_t^{t+T} \int_\Omega a|u_t|^2 dxds \right)^{1/2} \left(\int_t^{t+T} \|\nabla u\|^2 ds \right)^{1/2} \\ &+ C(E(t) + E(t+T)) + C \int_t^{t+T} \int_{\Gamma(x^0)} \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma ds \end{aligned} \tag{3.5}$$

with $\alpha > 0$, where we have used the Poincare's inequality.

Taking $N/2-1<\alpha<N/2$ and using (3.1) we have

$$\int_t^{t+T} E(s)ds \le C\{E(t+T)+D(t)^2\} + C\int_t^{t+T}\int_{\Gamma(x^0)} \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma ds. \tag{3.6}$$

To estimate the last term of the right-hand side of (3.6) we take a function $\eta \in C^1(\overline{\Omega})$ such that

$$0 \le \eta \le 1, \quad \eta = 1 \text{ on } \hat{\omega}, \quad \eta = 0 \text{ on } \Omega/\omega \quad \text{and} \quad |\nabla\eta|^2/\eta \in C(\overline{\Omega}), \tag{3.7}$$

where $\hat{\omega}$ is an open set in $\overline{\Omega}$ with $\Gamma(x_0) \subset \hat{\omega} \subset \omega$.

We multiply the equation by ηu and integrate to get

$$\begin{aligned} &\int_t^{t+T}\int_\Omega \eta|\nabla u|^2 dxds \\ &\le C(E(t)+E(t+T)) + C\int_t^{t+T}\int_\omega (|u_t|^2+|u|^2)dxds, \end{aligned} \tag{3.8}$$

where we have used the inequality

$$|(u, \nabla\eta\cdot\nabla u)| \le C|u||\sqrt{\eta}\nabla u|.$$

Further, we take an open set $\tilde{\omega}$ in R^N with $\tilde{\omega}\cap\overline{\Omega} \subset \hat{\omega}$ and a C^1 vector field h such that $h=\nu$ on $\Gamma(x^0), h\cdot\nu \ge 0$ on $\partial\Omega$ and $h=0$ on $\Omega/\tilde{\omega}$. Then, we multiply the equation by $h\cdot\nabla u$ and integrate to get

$$\begin{aligned} \int_t^{t+T}\int_{\Gamma(x^0)} |\frac{\partial u}{\partial \nu}|^2 dxds &\le \int_t^{t+T}\int_{\partial\Omega}(h\cdot\nu)|\frac{\partial u}{\partial \nu}|^2 dxds \\ &\le C\int_t^{t+T}\int_{\tilde{\omega}}(|u_t|^2+|\nabla u|^2)dxds + C(E(t+T)+E(t)). \end{aligned} \tag{3.9}$$

(See (3.4).)

From (3.6), (3.8) and (3.9) we obtain the estimate (3.3).

Since

$$TE(t+T) \le \int_t^{t+T} E(s)ds$$

we know from (3.3) that if we take $T>2C$,

$$E(t+T) \le C\{D(t)^2 + \int_t^{t+T}\int_\omega \{|u_t|^2+|u|^2\}\, dxds\}. \tag{3.10}$$

We take $T>0$ as above in the sequel.

We proceed to estimations of the last two terms of the right-hand side of (3.10). To treat the last term we prepare the following inequality, which is a variant of the estimate (1.12) in [17]. (See also (5.15) in [1].)

PROPOSITION: *We take a large $T > 0$. Then, there exists a constant $C > 0$ independent of (u_0, u_1) such that the estimate*

$$\int_t^{t+T} \|u(s)\|^2 ds \le C \left\{ \int_t^{t+T} \int_\Omega a(x)|u_t|^2 dx ds + \int_t^{t+T} \int_\omega |u_t|^2 dx ds \right\} \tag{3.11}$$

holds for any energy finite solutions of (1.1)–(1.2).

Proof: The proof is given quite similarly as in the proof of (1.12) in [17]. We sketch it briefly. Suppose that the assertion was false. Then, there would exist a sequence $\{t_n\} \subset R$ and a sequence of solutions $\{u_n\}$ such that

$$\int_{t_n}^{t_n+T} \|u_n(s)\|^2 ds = 1,$$

and

$$\int_{t_n}^{t_n+T} \int_\Omega a(x)|u_{nt}|^2 + \int_{t_n}^{t_n+T} \int_\omega |u_{nt}|^2 dx ds \to 0$$

as $n \to \infty$. We note that the inequality (3.10) remains valid by homogeneity even if we replace $u(t)$ by $u_n(t)$. Thus, setting $v_n(t) = u_n(t + t_n)$ we have

$$\int_0^T \|v_n(s)\|^2 ds = 1,$$

$$\int_0^T \int_\Omega a(x)|v_{nt}(s)|^2 dx ds + \int_0^T \int_\omega |v_{nt}(t)|^2 dx ds \to 0 \tag{3.12}$$

as $n \to \infty$, and, by (3.1) and (3.10),

$$\sup_{0 \le t \le T} \{\|v_{nt}(s)\|^2 + \|\nabla v_n(t)\|^2\} = 2E(v_n(0))$$

$$= 2\left\{ E(v_n(T)) + \int_0^T \int_\Omega a(x)|v_{nt}(s)|^2 dx ds \right\} \le C_1 < \infty$$

for large n, where C_1 is a constant independent of (u_0, u_1). Therefore, $\{v_n(t)\}$ converges along a subsequence to a function $v(t) \in C([0,T]; H_1^0) \cap C^1([0,T]; L^2)$ in appropriate topologies, which is a solution of the problem

$$v_{tt} - \Delta v = 0 \quad \text{in } \Omega \times [0,T] \quad (\text{ in fact in } \Omega \times R) \quad \text{and} \quad v|_{\partial\Omega} = 0$$

with the additional conditions

$$\int_0^T \int_\Omega a(x)|v_t|^2 dx ds = 0$$

and

$$\int_0^T \|v(s)\|^2 ds = 1.$$

This is a contradiction if we take a large $T > 0$ ($T > d(\Omega)$, diameter of Ω, is sufficient), because the first condition then implies $v \equiv 0$ for a sólution of the wave equation above.

Now, the inequality (3.10) together with Proposition 2 implies

$$E(t+T) \leq C\left\{D(t)^2 + \int_t^{t+T}\int_\omega |u_t|^2 dxds\right\}. \tag{3.13}$$

Finally, by the assumption on $a(x)$ and Lemma 1 we see (cf. [8])

$$\begin{aligned}
&\int_t^{t+T}\int_\omega |u_t|^2 dxds \\
&\leq \left\{\int_t^{t+T}\int_\omega a|u_t|^2 dxds\right\}^{p/(p+1)}\left\{\int_t^{t+T}\int_\omega a^{-p} dxds\right\}^{1/(p+1)} \\
&\quad \times \sup_{t\leq s\leq t+1} \|u_t(s)\|_\infty^{2/(p+1)} \\
&\leq CD(t)^{2p/(p+1)} \sup_{t\leq s\leq t+T} \|u_t(s)\|^{2(1-N/2m)/(p+1)}\|u_t(t)\|_{H_m}^{N/m(p+1)} \\
&\leq C_m D(t)^{2p/(p+1)} E(t)^{(2m-N)/2m(p+1)} \equiv A(t)^2 \qquad (3.14)
\end{aligned}$$

where $C_m = C(\|u_0\|_{H_{m+1}} + \|u_1\|_{H_m})^{N/m(p+1)}$. Thus, we have from (3.13)

$$E(t+T) \leq C(D(t)^2 + A(t)^2) \tag{3.15}$$

and, returning to the identity (3.1),

$$E(t) \leq C(D(t)^2 + A(t)^2). \tag{3.16}$$

Thus, recalling the definition of $A(t)^2$ and using Young's inequality we arrive at the inequality

$$E(t) \leq CD(t)^2 + C(\|u_0\|_{H_{m+1}} + \|u_1\|_{H_m})^{2N/(2mp+N)} D(t)^{4mp/(2mp+N)}, \tag{3.17}$$

or

$$E(t)^{1+N/2mp} \leq C(\|u_0\|_{H_{m+1}} + \|u_1\|_{H_M})^{N/mp}(E(t) - E(t+T)). \tag{3.18}$$

Now, applying Lemma 2 to the above inequality (3.18) we obtain the decay estimate (2.7). The proof of Theorem 1 is complete.

4. Proof of Theorem 2

In this section we treat the semilinear equation

$$u_{tt} - \Delta u + a(x)u_t + f(u) = 0 \quad \text{in } [0,\infty) \times \Omega, \tag{4.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{and} \quad u|_{\partial\Omega} = 0. \tag{4.2}$$

Let $m > N/2$ and $f \in C^m(R)$. Then, it is well known (cf. T. Kato [6]) that the problem (4.1)–(4.2) admits a unique local solution $u(t)$ on $[0,\tilde{T}) \times \Omega$ for some $\tilde{T}$ in the class

$$X_m(\tilde{T}) = \bigcap_{k=1}^{m} C^k([0,\tilde{T}); H_{m+1-k} \cap H_1^0) \cap C^{m+1}([0,\tilde{T}); L^2(\Omega)),$$

where (u_0, u_1) should satisfy the compatibility condition of the m th order.

Thus, for the proof of Theorem 2 it suffices to derive the a priori estimates

$$\sum_{k=0}^{m+1} \|D^k u(t)\| < K < \infty$$

and

$$E(t) \equiv \frac{1}{2}(\|u_t(t)\|^2 + \|\nabla u(t)\|^2) \le C_m(1+T)^{-2pm/N}$$

under the conditions that

$$\|u_0\|_{H_{m+1}} + \|u_1\|_{H_m} < K$$

and $I_0^2 \equiv \|\nabla u_0\|^2 + \|u_1\|^2$ is small, where we take in fact $N = 2, 3$ and $m = 4$.

Setting $v(t) = {}^t(u, u_t)$ the linear equation (1.1)–(1.2) is reduced to the system

$$v(t) = Av(t)$$

where

$$A = \begin{pmatrix} 0 & I \\ \Delta & -aI \end{pmatrix} \quad \text{with } D(A) = H_2 \cap H_1^0 \times H_1^0.$$

We denote by $U(t)$ the semi-group (in fact, group) generated by A. Then, Theorem 1 implies that

$$\begin{aligned} \|U(t)v_0\| \le g_K^{-pm/N}(t) &\equiv C(I_0^{-N/pm} + K^{-N/pm}(t-T)^+)^{-pm/N} \\ &\le CK(1+t)^{-pm/N} \end{aligned} \tag{4.3}$$

provided that $v_0 = (u_0, u_1)$ satisfies the compatibility condition of the mth order associated with (1.1)–(1.2) and $\|v_0\|_{H_{m+1}\times H_m} \le K$. By a standard theory of semi-groups the semilinear problem (4.1)–(4.2) is equivalent to

$$v(t) = U(t)v_0 + \int_0^t U(t-s)\tilde{f}(u(s))ds \tag{4.4}$$

where we set $\tilde{f}(u) = {}^t(0, f(u))$.

To derive the desired a priori estimates we assume that

$$\|u(t)\|_\infty \le L, \quad \sum_{k=0}^{m+1} \|D^k u(t)\| < K \quad \text{and} \quad E(t) \le M g_K^{-2pm/N}(t) \tag{4.5}$$

for $0 \le t < \tilde{T}$ with some $\tilde{T} > 0$, where M is a constant to be fixed later.

To apply (4.3) to the above integral equation we must check that for every fixed $t \ge 0, (\tilde{u}_0, \tilde{u}_1) \equiv (0, f(u(t)))$ satisfy the compatibility condition of the 4th order associated with the linear equation. Let $\{\tilde{u}_k\}$ be defined as $\{u_k\}$ in (2.2) with (u_0, u_1) replaced by $(\tilde{u_0}, \tilde{u}_1)$. Then, we see

$$\tilde{u}_2 = \Delta\tilde{u}_0 - a(x)\tilde{u}_1 = -af(u(t)) \in H_1^0$$

and

$$\tilde{u}_3 = \Delta\tilde{u}_1 - a(x)\tilde{u}_2 = \Delta(f(u(t))) + a^2 f(u) \in H_1^0$$

where we have used the facts that

$$f(0) = f'(0) = f''(0) = 0.$$

Further, we see

$$\tilde{u}_4 = \Delta\tilde{u}_2 - a\tilde{u}_3 = -\Delta\{af(u)\} - a\Delta(f(u)) - a^3 f(u) \in H_1^0$$

and

$$\tilde{u}_5 = \Delta\tilde{u}_3 - a\tilde{u}_4 = \Delta^2(f(u)) + \Delta(a^2 f(u)) - a\tilde{u}_4 \in L^2.$$

Moreover, by a similar argument we can show that $v_0 = (u_0, u_1)$, which originally satisfied the compatibility condition of 4th order associated with the semilinear equation, satisfies also the condition associated with the linear equation. Thus, we have

$$\|U(t-s)\tilde{f}(u(s))\| \le CK(1+t-s)^{-mp/N}\|f(u(s))\|_{H_m} \tag{4.6}$$

and

$$\|U(t)v_0\| \le g_K^{-pm/N}(t) \tag{4.7}$$

with $m = 4$.

Here, we can prove

$$\|f(u(s))\|_{H_m} \le C(M,K)g_K(s)^{-2mp/N} \quad (m = 4) \tag{4.8}$$

under the assumptions (4.5). This estimate is included essentially in [12] and [13]. For completeness, however, we sketch it briefly. We see

$$\begin{aligned}\|D^m f(u)\| \le & \|f'(u)D^m u\| + \left\| f''(u)\sum_{i=1}^{m-1} D^i u D^{m-1}u \right\| \\ & + \left\| \sum_{j=3}^{4} f^{(j)}(u) \sum_{\alpha \in S_j} (Du)^{\alpha_1}\cdots(D^j u)^{\alpha_j} \right\| \\ \equiv & I_1 + I_2 + I_3\end{aligned} \tag{4.9}$$

where

$$S_j = \left\{ (\alpha_1, \cdots, \alpha_j) \in N^j \,|\, \sum_{i=1}^{j} \alpha_i = j \text{ and } \sum_{i=1}^{j} i\alpha_i = m \right\}.$$

Here, by Lemma 1, we know

$$\begin{aligned} I_1 & \le C\left(\int_\Omega |u|^4 |D^m u|^2\right)^{\frac{1}{2}} \\ & \le C\|\nabla u\|^{3-\eta_1}\|D^{m+1}u\|^{\eta_1} \\ & \le C(M,K)g_K^{(3-\eta_1)pm/N}(s) \end{aligned} \tag{4.10}$$

with

$$\eta_1 = \frac{N}{m}\left(\frac{m-1}{N} - \frac{N-2}{2N}\right) = 1 - N/8 \le 1$$

(a trivial modification is needed if $N = 2$).

Similarly,

$$\begin{aligned} I_2 & \le C\|uD^i u D^{m-i}u\| \\ & \le C\|\nabla u\|^{3-\eta_2}\|D^{m+1}u\|^{\eta_2} \\ & \le C(M,K)g_K^{(3-\eta_2)pm/N}(s) \end{aligned} \tag{4.11}$$

with

$$\eta_2 = 1 - N/m < 1.$$

Finally,

$$\begin{aligned} I_3 &\le C \sum_{j=3}^{4} \sum_{\alpha \in S_j} \|Du\|_{p_1\alpha_1}^{\alpha_1} \cdots \|D^j u\|_{p_j\alpha_j}^{\alpha_j} \\ &\le C \sum_{j=3}^{4} \Pi_{i=1}^{j} \|Du\|^{\alpha_i(1-\theta_i)} \|D^{m+1}u\|^{\alpha_i\theta_i} \\ &\le C(M,K) g_K^{-pm\sum \alpha_i(1-\theta^i)/N}(s) \end{aligned} \tag{4.12}$$

with

$$\theta_i = \frac{N}{m}\left(\frac{i-1}{N} + \frac{1}{2} - \frac{1}{p_i\alpha_i}\right) \quad \text{and} \quad \sum_{i=1}^{j} \frac{1}{p_i} = \frac{1}{2} \quad (1 < p_i \le \infty).$$

It is easy to see

$$\sum_i \alpha_i \theta_i = \frac{m-j}{m} + \frac{Nj}{2m} - \frac{N}{2m}$$

and hence

$$I_3 \le C(M,K) g_K^{-p(2m+3-N)/N}(s). \tag{4.13}$$

Thus, we conclude (4.8).

From (4.4), (4.6) and (4.8) we have

$$\begin{aligned} \sqrt{E(t)} &\le \|v(t)\| \\ &\le g_K^{-pm/N}(t) + C(K,M) \int_0^t (1+t-s)^{-mp/N} g_K^{-2mp/N}(s)ds \\ &= g_K^{-pm/N}(t) + C(K,M) \left\{ \int_0^{t/2} + \int_{t/2}^{t} \right\} \\ &\le g_K^{-pm/N}(t) + CI_0^2(1+t)^{-pm/N} + CI_0(1+t)^{-2mp/N+1} g_K^{-pm/N}(t) \\ &\le (1 + C(K,M)I_0) g_K^{-pm/N}(t) \end{aligned} \tag{4.14}$$

provided that

$$2mp = 8p > N,$$

which is just our assumption on p.

Thus, we observe that if we take $M > 1$ and choose I_0 so small that

$$1 + C(K, M)I_0 < \sqrt{M} \tag{4.15}$$

may hold, then

$$E(t) < M g_K^{-2pm/N} \quad \text{for } 0 \le t < \infty \tag{4.16}$$

as long as

$$\sum_{j=0}^{m+1} \|D_t^j u(t)\|_{H_{m+1-j}} \le K \quad \text{and} \quad \|u(t)\|_\infty \le L.$$

We easily see, by Lemma 1,

$$\|u(t)\|_\infty \le C\|\nabla u(t)\|^{1-\theta}\|u(t)\|_{H_{m+1}}^\theta \le C I_0^{1-\theta} K^\theta$$

with

$$\theta = (N-2)/2m \quad \text{(arbitrary small positive number if } N = 2\text{)}.$$

Hence, we have

$$\|u(t)\|_\infty < L$$

provided that

$$C I_0^{1-\theta} K^\theta < L. \tag{4.17}$$

Finally, on the basis of (4.16), we shall derive the estimate

$$\sum_{j=0}^{m+1} \|D_t^j u(t)\|_{H_{m+1-j}} \le C(\|u_0\|_{H_{m+1}} + \|u_1\|_{H_m}) + q(K, I_0) \equiv Q(K, I_0), \tag{4.18}$$

as long as

$$\sum_{j=0}^{m+1} \|u(t)\|_{H_{m+1-j}} \le K,$$

where $q(K, I_0)$ is a quantity depending on K, I_0 in such a way that

$$\lim_{I_0 \to 0} q(K, I_0) = 0.$$

Let us begin with

$$E_m(t) \equiv \frac{1}{2}(\|D_t^{m+1}u(t)\|^2 + \|D_t^m D_x u(t)\|^2).$$

Differentiating the equation m times in t we get

$$D_t^{m+2}u(t) - \Delta D_t^m u(t) + a(x)D_t^{m+1}u(t) = D_t^m f(u). \tag{4.19}$$

We already know

$$\|D_t^m f(u(t))\| \le C(K) g_K^{-2pm/N}(t) \quad (m = 4)$$

and hence, multiplying (4.19) by $D_t^{m+1}u$, we have

$$\frac{d}{dt}E_m(t) \le \|D_t^m f(u(t))\| \sqrt{E_m(t)} \le C(K) g_K^{-2pm/N}(t) \sqrt{E_m(t)},$$

which yields

$$\begin{aligned} E_m(t) &\le \left\{ \sqrt{E_m(0)} + C \int_0^\infty g_K^{-2pm/N}(s) ds \right\}^2 \\ &\le C(\|u_0\|_{H_{m+1}}^2 + \|u_1\|_{H_m}^2) + C K^{2N/pm} I_0^4. \end{aligned} \tag{4.20}$$

Next, we use the equation

$$-\Delta D_t^{m-1}u(t) = -D_t^{m+1}u(t) - aD_t^m u(t) + D_t^{m-1}f(u) \equiv h(t). \tag{4.21}$$

Using the estimate (4.20) just obtained we see easily (cf. the proof of (4.8))

$$\begin{aligned} \|h(t)\| &\le \|D_t^{m+1}u(t)\| + C\|D_t^m u(t)\| + \|D_t^{m-1}f(u)\| \\ &\le C(\|u_0\|_{H_{m+1}} + \|u_1\|_{H_m}\|) + C(K) g_K^{-2pm/N}(t) \\ &\le C(\|u_0\|_{H_{m+1}} + \|u_1\|_{H_m}) + C(K) I_0^2, \end{aligned}$$

and, by elliptic theory,

$$\|D_t^{m-1}u(t)\|_{H_2} \le C(\|u_0\|_{H_{m+1}} + \|u_1\|_{H_m}) + q_1(K, I_0) \tag{4.22}$$

for a certain $q_1(K, I_0)$ with $\lim_{I_0 \to 0} q_1(K, I_0) = 0$. Repeating similar arguments inductively we can prove further

$$\sum_{j=0}^{m-2} \|D_t^j u(t)\|_{H_{m+1-j}} \le C(\|u_0\|_{H_{m+1}} + \|u_1\|_{H_m}) + q_2(K, I_0) \tag{4.23}$$

with $\lim_{I_0 \to 0} q_2(K, I_0) = 0$. Thus, we conclude (3.17).

The estimate (4.18) means that if we take $K > C(\|u_0\|_{H_{m+1}} + \|u_1\|_{H_m})$ and choose I_0 so small that, in addition to (4.15) and (4.17), the inequality

$$Q(K, I_0) < K$$

may hold, then the local solution $u(t)$ continues to exist in fact on $[0, \infty)$ and all the estimates obtained are valid on $[0, \infty)$. The proof of Theorem 2 is now complete.

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